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Higher

Mathematics

HSN24400

Course Revision Notes

Contents

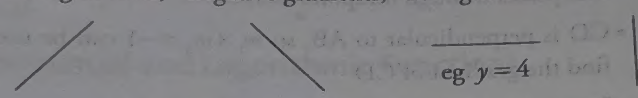
Straight Lines

Distance Formula

- Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ between points (x_1, y_1) and (x_2, y_2)

Gradients

- $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ between the points (x_1, y_1) and (x_2, y_2) where $x_1 \neq x_2$
- Positive gradients, negative gradients, zero gradients, undefined gradients



- Lines with the same gradient are parallel

  eg The line parallel to $2y+3x=5$

  has gradient $m=-\frac{3}{2}$ since $2y + 3x = 5$

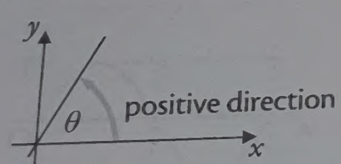
  $$2y = -3x + 5$$

  $$y = -\tfrac{3}{2}x + \tfrac{5}{2} \quad \text{(must be in the form } y = mx + c\text{)}$$

- Perpendicular lines have gradients such that $m \times m_{\text{perp.}} = -1$

  eg if $m = \frac{2}{3}$ then $m_{\text{perp.}} = -\frac{3}{2}$

- $m = \tan\theta$



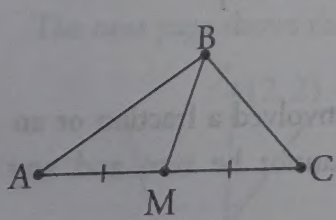
θ is the angle that the line makes with the positive direction of the x-axis

Equation of a Straight Line

- The line passing through (a, b) with gradient m has equation:

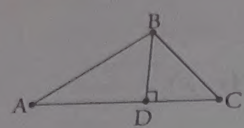
$$y - b = m(x - a)$$

Medians



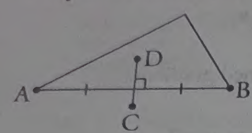
- M is the midpoint of AC, ie $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$
- BM is not usually perpendicular to AC, so $m_1 \times m_2 = -1$ cannot be used
- To work out the gradient of BM, use the gradient formula

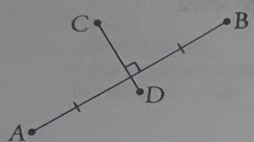
Altitudes



- D is not usually the midpoint of AC
- BD is perpendicular to AC, so $m_1 \times m_2 = -1$ can be used to work out the gradient of BD

Perpendicular Bisectors





- CD passes through midpoint of AC
- CD is perpendicular to AB, so $m_1 \times m_2 = -1$ can be used to find the gradient of CD
- Perpendicular bisectors do not necessarily have to appear within a triangle – they can occur with straight lines

Functions and Graphs

Composite Functions

Example

If $f(x) = x^2 - 2$ and $g(x) = \frac{1}{x}$, find a formula for

(a) $h(x) = f(g(x))$

(b) $k(x) = g(f(x))$

and state a suitable domain for each.

(a)
$$\begin{aligned} h(x) &= f(g(x)) \\ &= f\left(\tfrac{1}{x}\right) \\ &= \left(\tfrac{1}{x}\right)^2 - 2 \\ &= \frac{1}{x^2} - 2 \end{aligned}$$

Domain: $\{x : x \in \mathbb{R},\ x \neq 0\}$

(b)
$$\begin{aligned} k(x) &= g(f(x)) \\ &= g(x^2 - 2) \\ &= \frac{1}{x^2 - 2} \end{aligned}$$

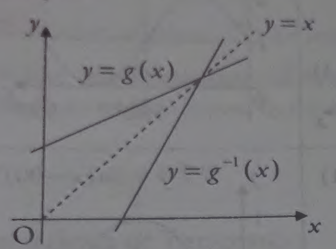
Domain: $\{x : x \in \mathbb{R},\ x \neq \pm\sqrt{2}\}$

- You will probably only be asked for a domain if the function involved a fraction or an even root. Remember that in a fraction the denominator cannot be zero and any number being square rooted cannot be negative

eg $f(x) = \sqrt{x+1}$ could have domain: $\{x : x \in \mathbb{R},\ x \geq -1\}$

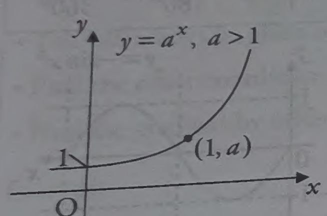
Graphs of Inverses

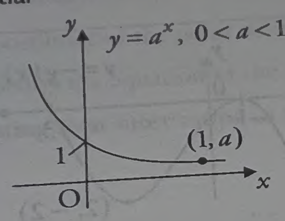
- To draw the graph of an inverse function, reflect the graph of the function in the line $y = x$



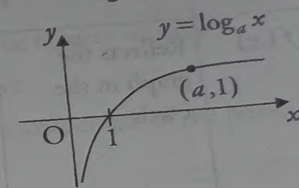
Exponential and Logarithmic Functions

Exponential



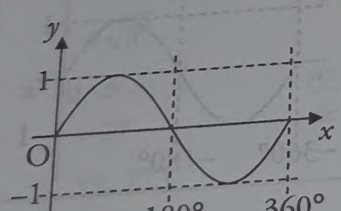


Logarithmic



Trigonometric Functions

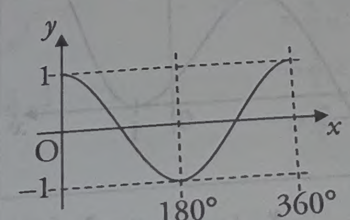
$y = \sin x$



Period $= 360°$

Amplitude $= 1$

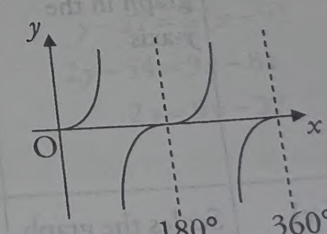
$y = \cos x$



Period $= 360°$

Amplitude $= 1$

$y = \tan x$

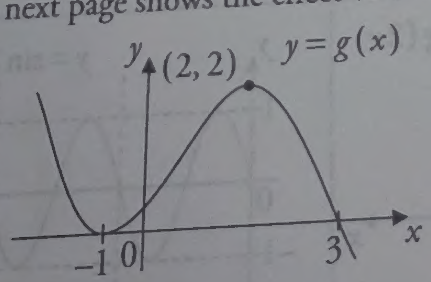


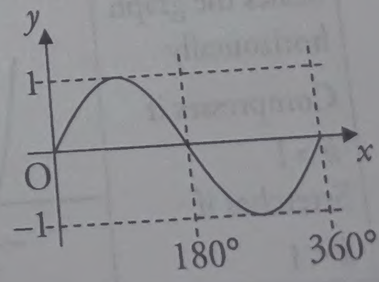
Period $= 180°$

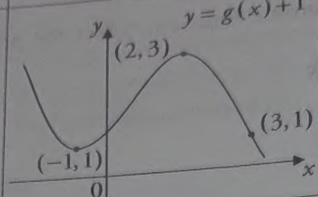
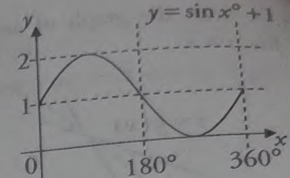
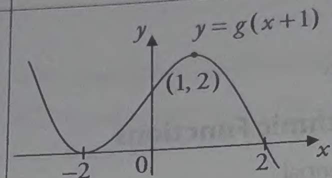
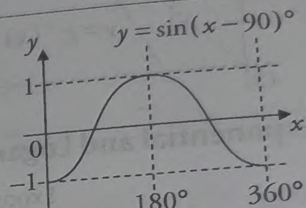
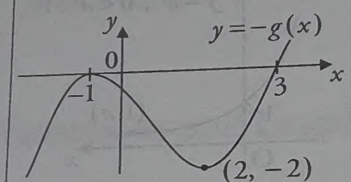
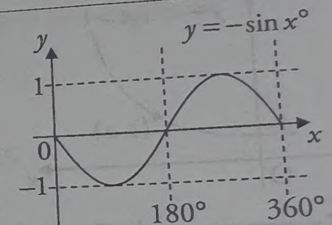
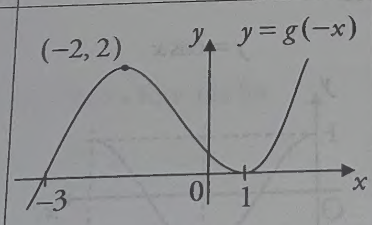
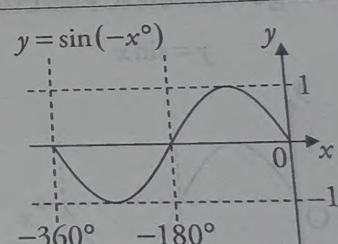
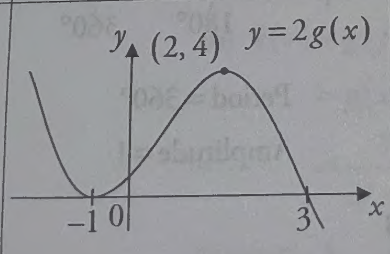
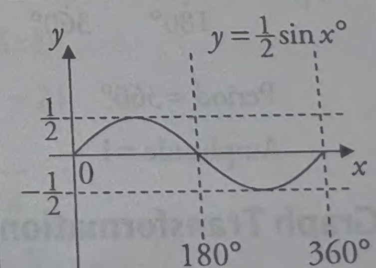
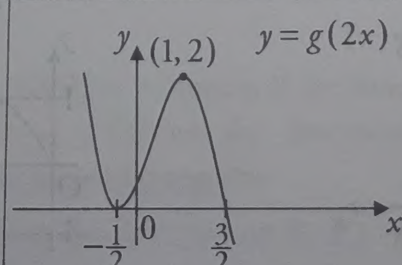
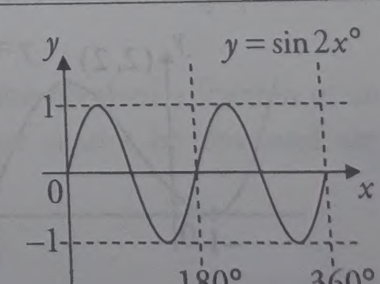
Amplitude is undefined

Graph Transformations

The next page shows the effect of transformations on the two graphs shown below.





Function	Effect	Effect on $f(x)$	Effect on $\sin x°$
$f(x)+a$	Shifts the graph a up the y-axis		
$f(x+a)$	Shifts the graph $-a$ along the x-axis		
$-f(x)$	Reflects the graph in the x-axis		
$f(-x)$	Reflects the graph in the y-axis		
$kf(x)$	Scales the graph vertically Stretches if $k>1$ Compresses if $k<1$		
$f(kx)$	Scales the graph horizontally Compresses if $k>1$ Stretches if $k<1$		

Differentiation

Differentiating

- If $f(x)=ax^n$ then $f'(x)=anx^{n-1}$
- Before you differentiate, all brackets should be multiplied out, and there should be no fractions with an x term in the denominator (bottom line), for example:

$$\frac{1}{3x^2}=\frac{1}{3}x^{-2} \qquad \frac{3}{x^2}=3x^{-2} \qquad \frac{1}{\sqrt{x}}=x^{-\frac{1}{2}}$$

Equations of Tangents

- Tangents are straight lines, therefore to find the equation of a tangent, you need a point on the line and its gradient to substitute into $y-b=m(x-a)$
- You will always be given one coordinate of the point which the tangent touches
- Find the other coordinate by solving the equation of the curve
- Find the gradient by differentiating then substituting in the x-coordinate of the point

Example

Find the equation of the tangent to the graph of $y=\sqrt{x^3}$ at the point where $x=9$.

$$\begin{aligned} y&=\sqrt{x^3}\\ &=\sqrt{9^3}\\ &=3^3\\ &=27 \end{aligned}$$

$(9,\ 27)$

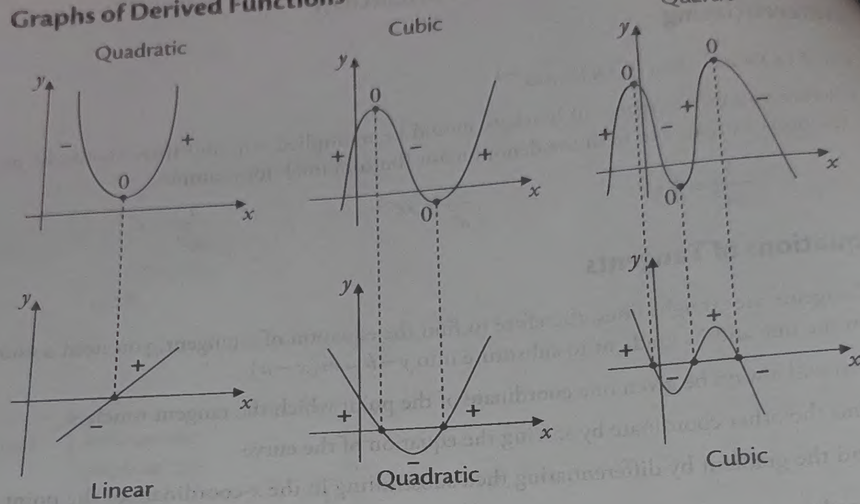
$$\begin{aligned} y&=\sqrt{x^3}\\ &=x^{\frac{3}{2}}\\ \frac{dy}{dx}&=\frac{3}{2}x^{\frac{1}{2}} \end{aligned}$$

At $x=9$, $m=\frac{3}{2}\times 9^{\frac{1}{2}}$
$$\begin{aligned} &=\tfrac{3}{2}\sqrt{9}\\ &=\tfrac{3}{2}\times 3\\ &=\tfrac{9}{2} \end{aligned}$$

$$\begin{aligned} y-b&=m(x-a)\\ y-27&=\tfrac{9}{2}(x-9)\\ 2y-54&=9x-81\\ 2y&=9x-27 \end{aligned}$$

- Stationary points occur at points where $\frac{dy}{dx}=0$
- You must justify the nature of turning points or points of inflection

Graphs of Derived Functions



Optimisation

- These types of questions are usually practical problems which involve maximum or minimum areas or volumes
- Remember you must show that a maximum or minimum exists

Sequences

Linear Recurrence Relations

- A linear recurrence relation is in the form $u_{n+1} = au_n + b$. Also be aware that this may be written as $u_n = au_{n-1} + b$
- If $-1 < a < 1$ then a limit $l = \dfrac{b}{1-a}$ exists. You must state this whenever you use the limit formula

Polynomials and Quadratics

Polynomials

- The degree of a polynomial is the value of the highest power, eg $3x^4+3$ has degree 4
- Synthetic division (nested form) can be used to factorise polynomials

Example

Find $\dfrac{4x^3-7x^2+11}{x+2}$.

-2	4	-7	0	11	Remember to put in 0 if there is no term
		-8	30	-60	
	4	-15	30	-49	

$$\frac{4x^3-7x^2+11}{x+2}=4x^2-15x+30 \text{ remainder } -49$$

ie $4x^3-7x^2+11=(x+2)(4x^2-15x+30)-49$

- If the divisor is a factor then the remainder is zero
- If the remainder is zero then the divisor is a factor

Completing the Square

- The x^2 term must have a coefficient of one. If it does not, you must take out a common factor from the x^2 and x term, but not the constant
- In the form $y=a(x+p)^2+q$ the turning point of the graph is $(-p,q)$

Example

Write $3x^2-12x+7$ in the form $a(x+p)^2+q$.

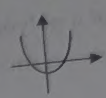
$$\begin{aligned}
&3x^2-12x+7\\
&=3(x^2-4x)+7\\
&=3(x^2-4x+(-2)^2-(-2)^2)+7\\
&=3((x-2)^2-4)+7\\
&=3(x-2)^2-12+7\\
&=3(x-2)^2-5
\end{aligned}$$

Note that in this example, the graph is ∪-shaped since the x^2 coefficient is positive; and the turning point is $(2,-5)$.

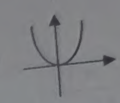
The Discriminant

- The discriminant is part of the quadratic formula and can be used to indicate how many roots a quadratic has. For the quadratic ax^2+bx+c:

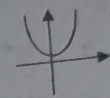
If $b^2-4ac>0$, the roots are real and unequal (distinct)



If $b^2-4ac=0$, the roots are real and equal (ie repeated roots)



If $b^2-4ac<0$, the roots are not real; they do not exist



- The discriminant can also be used to calculate the number of intersections between a line and a curve. To use it, you must first equate them and set equal to zero, before using the discriminant
- Remember if $b^2-4ac=0$, the line is a tangent

Integration

Integrating

- $\int ax^n \, dx = \dfrac{ax^{n+1}}{n+1}+c$
- As with differentiation, all brackets must be multiplied out, and there must be no fractions with an x term in the denominator

Examples

1. Find $\displaystyle\int \frac{dx}{\sqrt[8]{x^5}}$.

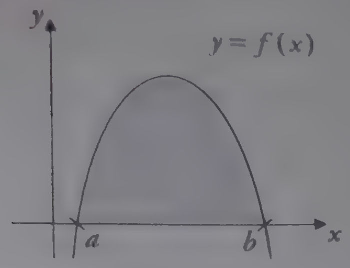
$$\int \frac{dx}{\sqrt[8]{x^5}} = \int \frac{1}{\sqrt[8]{x^5}}\,dx$$
$$= \int x^{-\frac{5}{8}}\,dx$$
$$= \frac{x^{\frac{3}{8}}}{\frac{3}{8}}+c$$
$$= \frac{8}{3}x^{\frac{3}{8}}+c$$
$$= \frac{8}{3}\sqrt[8]{x^3}+c$$

2. $\displaystyle\int \frac{x^2+5x^7}{x^2}\,dx$

$$\int \frac{x^2+5x^7}{x^2}\,dx = \int x^{-2}\left(x^2+5x^7\right)dx$$
$$= \int x^0+5x^5\,dx$$
$$= \int 1+5x^5\,dx$$
$$= x+\frac{5}{6}x^6+c$$

The Area under a Curve

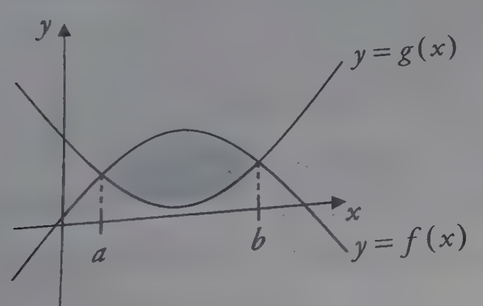
- If $F(x)$ is the integral of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$



- Remember that areas split by the x-axis must be calculated separately and any negative signs ignored; these just show that the area is under the axis.

The Area between two Curves

- The area between the graphs of $y = f(x)$ and $y = g(x)$ is defined as $\int_a^b f(x) - g(x)\,dx$



If the limits are not given, $f(x)$ and $g(x)$ should be equated to find a and b

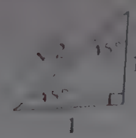
Trigonometry

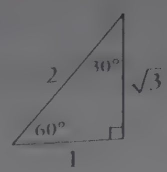
Background Knowledge

You should know how to use all of the information below:

- SOH CAH TOA
- $\tan x = \dfrac{\sin x}{\cos x}$
- $\sin^2 x + \cos^2 x = 1$
- The sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
- The cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

- The area of a triangle, $A = \frac{1}{2}ab\sin C$
- CAST diagrams
- Exact values:





Radians

- You should know how to convert between radians and degrees:

$360° = 2\pi$ $\quad 90° = \frac{\pi}{2}$ $\quad 45° = \frac{\pi}{4}$ $\quad$ Degrees $\xrightarrow{\div 180 \times \pi}$ Radians

$180° = \pi$ $\quad 60° = \frac{\pi}{3}$ $\quad 30° = \frac{\pi}{6}$ $\quad$ Radians $\xrightarrow{\times 180 \div \pi}$ Degrees

eg $\frac{5}{6}\pi = \frac{5 \times 180}{6} = 150°$

Trigonometric Equations

- Look at the restrictions on the domain, eg $0 \leq x° < 360$, or $0 \leq x < \pi$
- Be aware of whether the answer is required in degrees or radians
- Remember a CAST diagram whenever you are asked to "solve"

Examples

1. Solve $3\sin^2 x° = 1$ where $0 \leq x° < 360$.

$$3\sin^2 x° = 1$$
$$3(\sin x°)^2 = 1$$
$$(\sin x°)^2 = \frac{1}{3}$$
$$\sin x° = \pm\sqrt{\frac{1}{3}}$$
$$x° = \sin^{-1}\left(\pm\sqrt{\frac{1}{3}}\right)$$

✓S	A✓
✓T	C✓

$x° = 35.3°$ $\qquad x° = 180 - 35.3 = 144.7°$ $\qquad x° = 180 + 35.3 = 215.3°$ $\qquad x° = 360 - 35.3 = 324.7°$

Solution set $= \{35.3°, 144.7°, 215.3°, 324.7°\}$

Solve $2\sin 2x - 1 = 0$, $0 \le x < 2\pi$.

$$2\sin 2x - 1 = 0$$
$$2\sin 2x = 1$$
$$\sin 2x = \tfrac{1}{2}$$
$$2x = \sin^{-1}\left(\tfrac{1}{2}\right)$$

$$\begin{array}{c|c} \checkmark\ S & A\ \checkmark \\ \hline T & C \end{array}$$

$$2x^\circ = 30^\circ \qquad 2x^\circ = 180^\circ - 30^\circ$$
$$x^\circ = 15^\circ \qquad 2x^\circ = 150^\circ$$
$$x^\circ = 75^\circ$$

$$2x^\circ = 360^\circ + 30^\circ \qquad 2x^\circ = 360^\circ + 180^\circ - 30^\circ$$
$$2x^\circ = 390^\circ \qquad 2x^\circ = 510^\circ$$
$$x^\circ = 195^\circ \qquad x^\circ = 255^\circ$$

$$15^\circ = \frac{15}{180}\pi = \frac{3}{36}\pi = \frac{\pi}{12}$$
$$75^\circ = \frac{75}{180}\pi = \frac{15}{36}\pi = \frac{5}{12}\pi$$
$$195^\circ = \frac{195}{180}\pi = \frac{39}{36}\pi = \frac{13}{12}\pi$$
$$255^\circ = \frac{255}{180}\pi = \frac{51}{36}\pi = \frac{17}{12}\pi$$

$$\text{Solutions set} = \left\{\frac{\pi}{12}, \frac{5}{12}\pi, \frac{13}{12}\pi, \frac{17}{12}\pi\right\}$$

Compound Angle Formulae

- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- These are given on the formula sheet

Double Angle Formulae

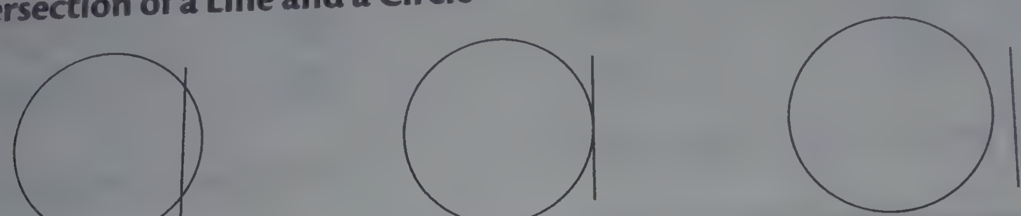
- $\sin 2A = 2\sin A \cos A$
- $\cos 2A = \cos^2 A - \sin^2 A$
  $= 1 - 2\sin^2 A$
  $= 2\cos^2 A - 1$
- These are given on the formula sheet

Circles

Equations of Circles

- A circle with centre (a, b) and radius r has the equation $(x-a)^2+(y-b)^2=r^2$
- Note that if a circle has centre $(0, 0)$ then the equation is $x^2+y^2=r^2$
- The equation can also be given in the form $x^2+y^2+2gx+2fy+c=0$ where the centre is $(-g, -f)$ and the radius $r=\sqrt{g^2+f^2-c}$
- You do not have to remember any of these equations, since they are all given in the exam
- You will have to remember the distance formula, $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, since this is not given, and is frequently used in circle questions

Intersection of a Line and a Circle



two intersections one intersection (tangency) no intersections

- Remember, a tangent and a line from the centre of a circle will meet at right angles, which means that $m_1 \times m_2 = -1$ can be used

Vectors

Basic Facts

- A vector is a quantity with both magnitude (size) and direction
- A vector is named either by using a directed line segment (eg $\overline{\text{AB}}$) or a bold letter (eg $\boldsymbol{u}$ written $\underline{u}$)
- A vector may also be defined in terms of $\underline{i}$, $\underline{j}$ and $\underline{k}$, the unit vectors in three perpendicular directions:

$$\underline{i}=\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \underline{j}=\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \underline{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

- The magnitude of vector $\overline{\text{AB}}=\begin{pmatrix}a\\b\end{pmatrix}$ is defined as $\left|\overline{\text{AB}}\right|=\sqrt{a^2+b^2}$

- $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{pmatrix}$
- $k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$ where k is a scalar
- Zero vector: $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- $\overrightarrow{OA}$ is called the position vector of the point A relative to the origin, written $\underset{\sim}{a}$
- $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$ where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are the position vectors of A and B
- If $\overrightarrow{AB} = k\overrightarrow{BC}$ where k is a scalar, then $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$. Since B is common to both $\overrightarrow{AB}$ and $k\overrightarrow{BC}$, then A, B and C are collinear

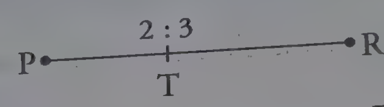
Dividing Vectors in a Ratio

- The point P can also be worked out from first principles, or
- Using the section formula. If P divides $\overrightarrow{AB}$ in the ratio $m:n$, then:

$$\underset{\sim}{p} = \frac{n}{m+n}\underset{\sim}{a} + \frac{m}{m+n}\underset{\sim}{b} \quad \text{where } \underset{\sim}{p} \text{ is the position vector } \overrightarrow{OP}$$

Example

P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$. Point T divides $\overrightarrow{PR}$ in the ratio 2:3. Work out the coordinates of point T.



Using the section formula

The ratio is 2:3, so let $m = 2$ and $n = 3$

$$\begin{aligned}
\underset{\sim}{t} &= \frac{n}{m+n}\underset{\sim}{p} + \frac{m}{m+n}\underset{\sim}{r} \\
&= \tfrac{3}{5}\underset{\sim}{p} + \tfrac{2}{5}\underset{\sim}{r} \\
&= \tfrac{1}{5}\left(3\underset{\sim}{p} + 2\underset{\sim}{r}\right) \\
&= \tfrac{1}{5}\left[\begin{pmatrix} -6 \\ 12 \\ -3 \end{pmatrix} + \begin{pmatrix} 16 \\ -2 \\ 38 \end{pmatrix}\right] \\
&= \tfrac{1}{5}\begin{pmatrix} 10 \\ 10 \\ 35 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix}
\end{aligned}$$

Therefore T is the point $(2, 2, 7)$.

From first principles

$$\begin{aligned}
\frac{\overrightarrow{PT}}{\overrightarrow{TR}} &= \frac{2}{3} \\
3\overrightarrow{PT} &= 2\overrightarrow{TR} \\
3\left(\underset{\sim}{t} - \underset{\sim}{p}\right) &= 2\left(\underset{\sim}{r} - \underset{\sim}{t}\right) \\
3\underset{\sim}{t} - 3\underset{\sim}{p} &= 2\underset{\sim}{r} - 2\underset{\sim}{t} \\
3\underset{\sim}{t} + 2\underset{\sim}{t} &= 2\underset{\sim}{r} + 3\underset{\sim}{p} \\
5\underset{\sim}{t} &= \begin{pmatrix} 16 \\ -2 \\ 38 \end{pmatrix} + \begin{pmatrix} -6 \\ 12 \\ -3 \end{pmatrix} \\
5\underset{\sim}{t} &= \begin{pmatrix} 10 \\ 10 \\ 35 \end{pmatrix} \\
\underset{\sim}{t} &= \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix}
\end{aligned}$$

Therefore T is the point $(2, 2, 7)$.

The Scalar Product

- The scalar product $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$, where θ is the smallest angle between $\underline{a}$ and $\underline{b}$
- Remember that both vectors must point away from the angle, eg

- If $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$
- $\cos\theta = \dfrac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$ or $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{|\underline{a}||\underline{b}|}$
- If $\underline{a}$ and $\underline{b}$ are perpendicular then $\underline{a}.\underline{b} = 0$
- If $\underline{a}.\underline{b} = 0$ then $\underline{a}$ and $\underline{b}$ are perpendicular

Example

If $\underline{u} = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$, calculate the angle between the vectors $\underline{u}+\underline{v}$ and $\underline{u}-\underline{v}$.

Let $\underline{a} = \underline{u} + \underline{v}$

$$\underline{a} = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{a} = \begin{pmatrix} 12 \\ 0 \\ 5 \end{pmatrix}$$

Let $\underline{b} = \underline{u} - \underline{v}$

$$\underline{b} = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{b} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$$

$$= \frac{(12\times4)+(0\times0)+(5\times3)}{\sqrt{12^2+0^2+5^2}\sqrt{4^2+0^2+3^2}}$$

$$= \frac{63}{\sqrt{169}\sqrt{25}}$$

$$\theta = \cos^{-1}\left(\frac{63}{\sqrt{169}\sqrt{25}}\right)$$

$$= 14{\cdot}3^\circ$$

Further Calculus

Trigonometry

Differentiation

- This is straightforward, since the formulae are given on the formula sheet:

$f(x)$	$f'(x)$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$

Integration

- Again, the formulae are provided in the paper:

$f(x)$	$\int f(x)\,dx$
$\sin ax$	$-\frac{1}{a}\cos ax + c$
$\cos ax$	$\frac{1}{a}\sin ax + c$

Examples

1. Differentiate $x^3 + \cos 3x$ with respect to x.

$$\tfrac{d}{dx}\left(x^3 + \cos 3x\right) = 3x^2 - 3\sin 3x$$

2. Find $\int 4x^3 + \sin 3x\,dx$.

$$\begin{aligned}\int 4x^3 + \sin 3x\,dx &= \frac{4x^4}{4} - \tfrac{1}{3}\cos 3x + c\\ &= x^4 - \tfrac{1}{3}\cos 3x + c\end{aligned}$$

Chain Rule Differentiation

- If $f(x) = (ax+b)^n$ then $f'(x) = n(ax+b)^{n-1} \times a = an(ax+b)^{n-1}$

or

- If $f(x) = (p(x))^n$ then $f'(x) = n(p(x))^{n-1} \times p'(x)$
- "The power multiplies to the front, the bracket stays the same, the power lowers by one and everything is multiplied by the differential of the bracket"

Examples

1. Given $f(x)=\frac{1}{x^2}+\sqrt{x}-\sin 3x$, find $f'(x)$.

$$f(x)=x^{-2}+x^{\frac{1}{2}}-\sin 3x$$
$$f'(x)=-2x^{-3}+\tfrac{1}{2}x^{-\frac{1}{2}}-3\cos 3x$$
$$=-\frac{2}{x^3}+\frac{1}{2\sqrt{x}}-3\cos 3x$$

2. Given $f(x)=(3x^2+2x+1)^3$, find $f'(x)$.

$$f'(x)=3(3x^2+2x+1)^2\times(6x+2)$$
$$=3(6x+2)(3x^2+2x+1)^2$$

3. Differentiate $y=\cos^2 x=(\cos x)^2$ with respect to x.

$$\frac{dy}{dx}=2(\cos x)\times(-\sin x)$$
$$=-2\cos x\sin x$$

Integration of $(ax+b)^n$

- $\int(ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{(n+1)\times a}+c$

Example

Find $\int(3x+5)^4\,dx$.

$$\int(3x+5)^4\,dx=\frac{(3x+5)^5}{5\times 3}+c=\frac{(3x+5)^5}{15}+c$$

- It is possible for any type of 'further calculus' to be examined in the style of a standard calculus question (eg optimisation, area under a curve, etc)

Exponentials and Logarithms

- An exponential is a function in the form $f(x)=a^x$
- Logarithms and exponentials are inverses
- $y=a^x \Leftrightarrow \log_a y=x$
- On a calculator, $\boxed{\log}$ is $\log_{10}$ and $\boxed{\ln}$ is $\log_e$

Laws of Logarithms

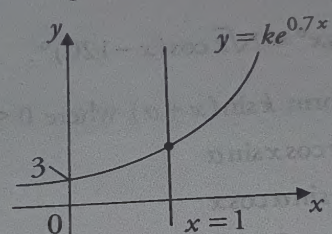
- $\log_a x + \log_a y = \log_a xy$ (Squash)
- $\log_a x - \log_a y = \log_a \frac{x}{y}$ (Split)
- $\log_a x^n = n\log_a x$ (Fly)

Examples

1. Evaluate $\log_2 4 + \log_2 6 - \log_2 3$

$$\log_2 4 + \log_2 6 - \log_2 3$$
$$= \log_2\left(\frac{4\times 6}{3}\right)$$
$$= \log_2 8$$
$$= 3 \quad (\text{since } 2^3 = 8)$$

2. Below is a diagram of part of the graph of $y = ke^{0.7x}$



(a) Find the value of k

(b) The line with equation $x = 1$ intersects at R. Find the coordinates of R.

(a) At $(0, 3)$, $y = ke^{0.7x}$

$$3 = ke^{0.7\times 0}$$
$$3 = ke^{0}$$
$$k = 3$$

(b) $x = 1 \Rightarrow y = 3e^{0.7\times 1}$

$$= 6.04$$

So R is the point $(1, 6.04)$.

The Wave Function

Example

1. Express $\sqrt{6}\sin x^\circ - \sqrt{2}\cos x^\circ$ in the form $k\cos(x-a)^\circ$ where $0 \le a^\circ < 360$.

$$k\cos(x-a)^\circ = k\cos x^\circ \cos a^\circ + k\sin x^\circ \sin a^\circ$$
$$= k\cos a^\circ \cos x^\circ + k\sin a^\circ \sin x^\circ$$

$k\cos a^\circ = -\sqrt{2}$

$k\sin a^\circ = \sqrt{6}$

$$\begin{array}{c|c} \checkmark\checkmark\ S & A\ \checkmark \\ \hline \checkmark\ T & C \end{array}$$

$$k = \sqrt{\left(-\sqrt{2}\right)^2 + \sqrt{6}^2} = \sqrt{2+6} = \sqrt{8} = 2\sqrt{2}$$

$$\tan a^\circ = \frac{k\sin a^\circ}{k\cos a^\circ} = -\frac{\sqrt{6}}{\sqrt{2}} = -\sqrt{3}$$

$$a^\circ = 180^\circ - \tan^{-1}\left(\sqrt{3}\right) = 180^\circ - 60^\circ = 120^\circ$$

Therefore $\sqrt{6}\sin x^\circ - \sqrt{2}\cos x^\circ = 2\sqrt{2}\cos(x-120)^\circ$.

2. Express $\cos x - \sin x$ in the form $k\sin(x+\alpha)$ where $0 \le \alpha < 2\pi$.

$$k\sin(x+\alpha) = k\sin x\cos\alpha + k\cos x\sin\alpha$$
$$= k\cos\alpha\sin x + k\sin\alpha\cos x$$

$k\cos\alpha = -1$

$k\sin\alpha = 1$

$$\begin{array}{c|c} \checkmark\checkmark\ S & A\ \checkmark \\ \hline \checkmark\ T & C \end{array}$$

$$k = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\tan\alpha = \frac{k\sin\alpha}{k\cos\alpha} = -1$$

$$\alpha^\circ = 180^\circ - \tan^{-1}(1) = 180^\circ - 45^\circ = 135^\circ$$

$$\alpha = \frac{135}{180}\pi = \frac{3}{4}\pi$$

Therefore $\cos x - \sin x = \sqrt{2}\sin\left(x + \frac{3}{4}\pi\right)$.

- The maximum value of an expression in the form $k\cos(x \pm a)$ occurs when $\cos(x \pm a) = 1$; and $\sin(x \pm a) = 1$ for $k\sin(x \pm a)$
- The minimum value of an expression in the form $k\cos(x \pm a)$ occurs when $\cos(x \pm a) = -1$; and $\sin(x \pm a) = -1$ for $k\sin(x \pm a)$